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The Relativistic Electrodynamics Least Action Principles Revisited: New Charged Point Particle and Hadronic String Models Analysis

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Abstract The classical relativistic least action principle is revisited from the vacuum field theory approach. New physically motivated versions of relativistic Lorentz type forces are derived, a new relativistic hadronic string model is proposed and analyzed in detail. The reasonings of R. Feynman, who argued that the relativistic dynamical expressions obtain true physical sense only with respect to the proper rest reference frames, are supported by analyzing the dynamical stability of a relativistic charged string model.

Keywords Relativistic electrodynamics · Least action principle · Lagrangian and Hamiltonian formalisms · Lorentz force · Hadronic string model

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1 Introduction

1.1 The Classical Relativistic Electrodynamics Backgrounds: A Charged Point Particle Analysis

It is commonly considered that classical electrodynamics is as the most fundamental physical theory, largely owing to the depth of its theoretical foundations and wealth of experimental verifications. Nowadays, in spite of the breadth and depth of theoretical understanding of electromagnetics, there remain several fundamental open problems and gaps in comprehension related to the true physical nature of Maxwell's theory when it comes to describing electromagnetic waves as quantum photons in a vacuum. These start with the difficulties in constructing a successful Lagrangian approach to classical electrodynamics that is free of the Dirac-Fock-Podolsky inconsistency [10, 12, 50] and end with the problem of devising its true quantization theory without such artificial constructions as a Fock space with indefinite metrics, the Lorentz condition in "average", and regularized "infinities" [10] of S-matrices. Moreover, there are the related problems of obtaining a complete description of the structure of a vacuum medium carrying the electromagnetic waves, and deriving a theoretically and physically valid Lorentz force expression for a moving charged point particle, possessing internal structure, and interacting with external electromagnetic field. It is well known [4, 26, 38, 50] that the relativistic least action principle for a point charged particle q in the Minkovski space-time $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$ can be formulated on a time interval $[t_1, t_2] \subset \mathbb{R}$ (in the light speed units) as

$$\begin{aligned} \delta S^{(t)} = 0, \quad S^{(t)} &:= \int_{\tau(t_1)}^{\tau(t_2)} (-m_0 d\tau - q\langle \mathcal{A}, dx\rangle_{M^4}) \\ &= \int_{s(t_1)}^{s(t_2)} (-m_0\langle \dot{x}, \dot{x}\rangle_{M^4}^{1/2} - q\langle \mathcal{A}, \dot{x}\rangle_{M^4}) ds. \end{aligned} \tag{1.1}$$

Here $\delta x(s(t_1)) = 0 = \delta x(s(t_2))$ are the boundary constraints, $m_0 \in \mathbb{R}_+$ is the so-called particle rest mass, the 4-vector $x := (t, r) \in M^4$ is the particle location in M^4, $\dot{x} := dx/ds \in T(M^4)$ is the particle 4-vector velocity with respect to a laboratory reference system $\mathcal{K}$, parameterized by means of the Minkovski space–time parameters $(s(t), r) \in M^4$ and related to each other by means of the infinitesimal Lorentz interval relationship

$$d\tau := \langle dx, dx\rangle_{M^4}^{1/2} := ds\langle \dot{x}, \dot{x}\rangle_{M^4}^{1/2}, \tag{1.2}$$

$\mathcal{A} \in T^*(M^4)$ is an external electromagnetic 4-vector potential, satisfying the classical Maxwell equations [26, 38, 50], the sign $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ means, in general, the corresponding scalar product in a finite-dimensional vector space $\mathcal{H}$ and $T(M^4), T^*(M^4)$ are, respectively, the tangent and cotangent spaces [1, 2, 22, 32, 56] to the Minkovski space M^4. In particular, $\langle x, x\rangle_{M^4} := t^2 - \langle r, r\rangle_{\mathbb{E}^3}$ for any $x := (t, r) \in M^4$.

The subintegral expression in (1.1)

$$\mathcal{L}^{(t)} := -m_0\langle \dot{x}, \dot{x}\rangle_{M^4}^{1/2} - q\langle \mathcal{A}, \dot{x}\rangle_{M^4} \tag{1.3}$$

is the related Lagrangian function, whose first part is proportional to the particle world line length with respect to the proper rest reference system $\mathcal{K}_r$ and the second part is proportional to the pure electromagnetic particle–field interaction with respect to the Minkovski laboratory reference system $\mathcal{K}$. Moreover, the positive rest mass parameter $m_0 \in \mathbb{R}_+$ is introduced into (1.3) as an external physical ingredient, also describing the point particle with respect to the proper rest reference system $\mathcal{K}_r$. The electromagnetic 4-vector potential $\mathcal{A} \in T^*(M^4)$ is at the same time expressed as a 4-vector, constructed and measured with respect to the Minkovski laboratory reference system $\mathcal{K}$ that looks, from physical point of view, enough controversial, since the action functional (1.1) is forced to be extremal with respect to the laboratory reference system $\mathcal{K}$. This, in particular, means that the real physical motion of our charged point particle, being realized with respect to the proper rest reference system $\mathcal{K}_r$, somehow depends on an external observation data [17, 19, 23–25, 27–29, 31, 33, 39–42, 44, 45, 47, 58, 59] with respect to the occasionally chosen laboratory reference system $\mathcal{K}$. This aspect was never discussed in the physical literature except for very interesting reasonings by R. Feynman in [26], who argued that the relativistic expression for the classical Lorentz force has a physical sense only with respect to the Euclidean rest reference system $\mathcal{K}_r$ variables $(\tau, r) \in \mathbb{E}^4$ related to the Minkovski laboratory reference system $\mathcal{K}$ parameters $(r, t) \in M^4$ by means of the infinitesimal relationship

$$d\tau := \langle dx, dx\rangle_{M^4}^{1/2} = dt(1-u^2)^{1/2}, \tag{1.4}$$

where $u := dr/dt \in T(\mathbb{E}^3)$ is the point particle velocity with respect to the reference system $\mathcal{K}$.

It is worth to point out here that to be correct, it would be necessary to include into the action functional the additional part describing the electromagnetic field itself. But this part is not taken into account, since there is generally assumed [7, 8, 18, 26, 36–38, 60] that the charged particle influence on the electromagnetic field is negligible. This is true, if the particle charge value q is very small but the support $supp\mathcal{A} \subset M^4$ of the electromagnetic 4-vector potential is compact. It is clear that in the case of two interacting with each other charged particles the above assumption cannot be applied, as it is necessary to take into account the relative motion of two particles and the respectively changing interaction energy. This aspect of the action functional choice problem appears to be very important when one tries to analyze the related Lorentz type forces exerted by charged particles on themselves. We will return to this problem in a separate section below.

Having calculated the least action condition (1.1), we easily obtain from (1.3) the classical relativistic dynamical equations

$$\begin{aligned} dP/ds &:= -\partial\mathcal{L}^{(t)}/\partial x = -q\nabla_x\langle\mathcal{A}, \dot{x}\rangle_{M^4}, \\ P &:= -\partial\mathcal{L}^{(t)}/\partial\dot{x} = m_0\dot{x}\langle\dot{x}, \dot{x}\rangle_{M^4}^{-1/2} + q\mathcal{A}, \end{aligned} \tag{1.5}$$

where by $P \in T^*(M^4)$ we denoted the common particle–field momentum of the interacting system.

Now at $s = t \in \mathbb{R}$ by means of the standard infinitesimal change of variables (1.4), we can easily obtain from (1.5) the classical Lorentz force expression

$$dp/dt = qE + qu \times B \tag{1.6}$$

with the relativistic particle momentum and mass

$$p := mu, \quad m := m_0(1-|u|^2)^{-1/2}, \quad |u|^2 := \langle u, u\rangle_{\mathbb{E}^3}, \tag{1.7}$$

respectively, the electric field

$$E := -\partial A/\partial t - \nabla\varphi \tag{1.8}$$

and the magnetic field

$$B := \nabla \times A, \tag{1.9}$$

where we have expressed the electromagnetic 4-vector potential as $\mathcal{A} := (\varphi, A) \in T^*(M^4)$.

The Lorentz force (1.6), owing to our preceding assumption, means the force exerted by the external electromagnetic field on our charged point particle, whose charge q is so negligible that it does not exert the influence on the field. This fact becomes very important if we try to make use of the Lorentz force expression (1.6) for the case of two charged interacting with each other particles, since then one cannot assume that our charge q exerts negligible influence on other charged particle. Thus, the corresponding Lorentz force between two charged particles should be suitably altered. Nonetheless, the modern physics [3, 6, 9–11, 16, 20, 21, 34, 35, 38, 48, 49, 55] did not make this naturally needed Lorentz force modification and there is used the classical expression (1.6). This situation was observed and analyzed concerning the related physical aspects in [54], having shown that the electromagnetic Lorentz force between two moving charged particles can be modified in such a way that it ceases to be dependent on their relative motion contrary to the classical relativistic case.

To our regret, the least action principle approach to analyze the Lorentz force structure was in [54] completely ignored and that gave rise to some incorrect and physically not motivated statements concerning mathematical physics backgrounds of the modern electrodynamics. To make the problem more transparent we will analyze it in the section below from the vacuum field theory approach recently devised in [12–15, 52].

1.2 The Least Action Principle Analysis

Consider the least action principle (1.1) and observe that the extremality condition

$$\delta S^{(t)} = 0, \qquad \delta x(s(t_1)) = 0 = \delta x(s(t_2)), \tag{1.10}$$

is calculated with respect to the laboratory reference system $\mathcal{K}$, whose point particle coordinates $(t, r) \in M^4$ are parameterized by means of an arbitrary parameter $s \in \mathbb{R}$ owing to expression (1.2). Recalling now the definition of the invariant proper rest reference system $\mathcal{K}_r$ time parameter (1.4), we obtain that at the critical parameter value $s = \tau \in \mathbb{R}$ the action functional (1.1) on the fixed interval $[\tau_1, \tau_2] \subset \mathbb{R}$ turns into

$$S^{(t)} = \int_{\tau_1}^{\tau_2} (-m_0 - q\langle \mathcal{A}, \dot{x}\rangle_{M^4}) d\tau \tag{1.11}$$

under the additional constraint

$$\langle \dot{x}, \dot{x}\rangle_{M^4}^{1/2} = 1, \tag{1.12}$$

where, by definition, $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$.

The expressions (1.11) and (1.12) need some comments since the corresponding (1.11) Lagrangian function

$$\mathcal{L}^{(t)} := -m_0 - q\langle \mathcal{A}, \dot{x}\rangle_{M^4} \tag{1.13}$$

depends virtually only on the unobservable rest mass parameter $m_0 \in \mathbb{R}_+$ and, evidently, it has no direct impact on the resulting particle dynamical equations following from the condition $\delta S^{(t)} = 0$. Nonetheless, the rest mass springs up as a suitable Lagrangian multiplier owing to the imposed constraint (1.12). To demonstrate this, consider the extended Lagrangian function (1.13) in the form

$$\mathcal{L}_\lambda^{(t)} := -m_0 - q\langle \mathcal{A}, \dot{x}\rangle_{M^4} - \lambda(\langle \dot{x}, \dot{x}\rangle_{M^4}^{1/2} - 1), \tag{1.14}$$

where $\lambda \in \mathbb{R}$ is a suitable Lagrangian multiplier. The resulting Euler equations look as follows

$$\begin{aligned} P_r &:= \partial\mathcal{L}_\lambda^{(t)}/\partial\dot{r} = qA + \lambda\dot{r}, \qquad P_t := \partial\mathcal{L}_\lambda^{(t)}/\partial\dot{t} = -q\varphi - \lambda\dot{t}, \\ \partial\mathcal{L}_\lambda^{(t)}/\partial\lambda &= \langle \dot{x}, \dot{x}\rangle_{M^4}^{1/2} - 1 = 0, \qquad dP_r/d\tau = q\nabla_r\langle A, \dot{r}\rangle_{\mathbb{E}^3} - q\dot{t}\nabla_r\varphi, \\ dP_t/d\tau &= q\langle \partial A/\partial t, \dot{r}\rangle_{\mathbb{E}^3} - q\dot{t}\partial\varphi/\partial t, \end{aligned} \tag{1.15}$$

giving rise, owing to relationship (1.4), to the following dynamical equations:

$$\frac{d}{dt}(\lambda u\dot{t}) = qE + qu \times B, \qquad \frac{d}{dt}(\lambda\dot{t}) = q\langle E, u\rangle_{\mathbb{E}^3}, \tag{1.16}$$

where we denoted by

$$E := -\partial A/\partial t - \nabla\varphi, \qquad B = \nabla \times A \tag{1.17}$$

the corresponding electric and magnetic fields. As a simple consequence of (1.16) one obtains

$$\frac{d}{dt}\ln(\lambda\dot{t}) + \frac{d}{dt}\ln(1-u^2)^{1/2} = 0, \tag{1.18}$$

being equivalent for all $t \in \mathbb{R}$, owing to relationship (1.4), to the relationship

$$\lambda\dot{t}(1-|u|^2)^{1/2} = \lambda := m_0, \tag{1.19}$$

where $m_0 \in \mathbb{R}_+$ is a constant, which could be interpreted as the rest mass of our charged point particle q. Really, the first equation of (1.16) can be rewritten as

$$dp/dt = qE + qu \times B, \tag{1.20}$$

where we denoted

$$p := mu, \quad m := \lambda\dot{t} = m_0(1-|u|^2)^{-1/2}, \tag{1.21}$$

coinciding exactly with that of (1.4).

Thereby, we retrieved here all the results obtained in the section above, making use of the action functional (1.11), expressed with respect to the rest reference system $\mathcal{K}_r$ under constraint (1.12). During these derivations, we were faced with a very delicate inconsistency property of definition of the action functional $S^{(t)}$, defined with respect to the rest reference system $\mathcal{K}_r$, but depending on the external electromagnetic potential function $\mathcal{A}: M^4 \to T^*(M^4)$, constructed exceptionally with respect to the laboratory reference system $\mathcal{K}$. Namely, this potential function, as a physical observable quantity, is defined and, respectively, measurable only with respect to the fixed laboratory reference system $\mathcal{K}$. This, in particular, means that a physically reasonable action functional should be constructed by means of an expression strongly calculated within the laboratory reference system $\mathcal{K}$ by means of coordinates $(t, r) \in M^4$ and later suitably transformed subject to the rest reference system $\mathcal{K}_r$ coordinates $(\tau, r) \in \mathbb{E}^4$, respective for the real charged point particle q motion. Thus, the corresponding action functional, in reality, should be from the very beginning written as

$$S^{(\tau)} = \int_{t(\tau_1)}^{t(\tau_2)} (-q\langle \mathcal{A}, \dot{x}\rangle_{M^4})dt, \tag{1.22}$$

where $\dot{x} := dx/dt$, $t \in \mathbb{R}$, being calculated on some time interval $[t(\tau_1), t(\tau_2)] \subset \mathbb{R}$, suitably related with the proper motion of the charged point particle q on the true time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the rest reference system $\mathcal{K}_r$ and whose charge value is assumed so negligible that it exerts no influence on the external electromagnetic field. Now the problem arises: how to compute correctly the variation $\delta S^{(\tau)} = 0$ of the action functional (1.22)?

To reply to this question we will turn to the Feynman reasonings from [26], where he argued, when deriving the relativistic Lorentz force expression, that the real charged particle dynamics can be determined physically not ambiguously only with respect to the rest reference system time parameter. Namely, Feynman wrote: "...we calculate a growth Δx for a small time interval Δt. But in the other reference system the interval Δt may correspond to changing both t' and x', thereby at the change of the only t' the suitable change of x will be other ... Making use of the quantity $d\tau$ one can determine a good differential operator $d/d\tau$, as it is invariant with respect to the Lorentz reference systems of transformations". This means that if our charged particle q moves in the Minkovski space M^4 during the time interval $[t_1, t_2] \subset \mathbb{R}$ with respect to the laboratory reference system $\mathcal{K}$, its proper real and invariant time of motion with respect to the rest reference system $\mathcal{K}_r$ will be respectively $[\tau_1, \tau_2] \subset \mathbb{R}$.

As a corollary of the Feynman reasonings, we arrive at the necessity to rewrite the action functional (1.22) as

$$S^{(\tau)} = \int_{\tau_1}^{\tau_2} (-q\langle \mathcal{A}, \dot{x}\rangle_{M^4})d\tau, \qquad \delta x(\tau_1) = 0 = \delta x(\tau_2), \tag{1.23}$$

where $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$, under the additional constraint

$$\langle \dot{x}, \dot{x}\rangle_{M^4}^{1/2} = 1, \tag{1.24}$$

being equivalent to the infinitesimal transformation (1.4). Simultaneously the proper time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ is mapped on the time interval $[t_1, t_2] \subset \mathbb{R}$ by means of the infinitesimal transformation

$$dt = d\tau(1 + |\dot{r}|^2)^{1/2}, \tag{1.25}$$

where $\dot{r} := dr/d\tau$, $\tau \in \mathbb{R}$. Thus, we can now pose the true least action problem equivalent to (1.23) as

$$\delta S^{(\tau)} = 0, \qquad \delta r(\tau_1) = 0 = \delta r(\tau_2), \tag{1.26}$$

where the functional

$$S^{(\tau)} = \int_{\tau_1}^{\tau_2} [-\bar{W}(1 + |\dot{r}|^2)^{1/2} + q\langle A, \dot{r}\rangle_{\mathbb{E}^3}]d\tau \tag{1.27}$$

is characterized by the Lagrangian function

$$\mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r}|^2)^{1/2} + q\langle A, \dot{r}\rangle_{\mathbb{E}^3}. \tag{1.28}$$

Here we denoted, for further convenience, $\bar{W} := q\varphi$, being the suitable vacuum field [12, 51, 52, 54] potential function. The resulting Euler equation gives rise to the following relationships

$$\begin{aligned} P &:= \partial \mathcal{L}^{(\tau)}/\partial \dot{r} = -\bar{W}\dot{r}(1+|\dot{r}|^2)^{-1/2} + qA, \\ dP/d\tau &:= \partial \mathcal{L}^{(\tau)}/\partial r = -\nabla \bar{W}(1+|\dot{r}|^2)^{1/2} + q\nabla\langle A, \dot{r}\rangle_{\mathbb{E}^3}. \end{aligned} \tag{1.29}$$

Now making use once more of the infinitesimal transformation (1.25) and the crucial dynamical particle mass definition [12, 51, 52, 54, 57] (in the light speed units)

$$m := -\bar{W}, \tag{1.30}$$

we can easily rewrite equations (1.29) with respect to the parameter $t \in \mathbb{R}$ as the classical relativistic Lorentz force:

$$dp/dt = qE + qu \times B, \tag{1.31}$$

where we denoted

$$\begin{aligned} &p := mu, \quad u := dr/dt, \\ &B := \nabla \times A, \qquad E := -q^{-1}\nabla \bar{W} - \partial A/\partial t. \end{aligned} \tag{1.32}$$

Thus, we obtained once more the relativistic Lorentz force expression (1.31), but slightly different from (1.6), since the classical relativistic momentum of (1.7) does not completely coincide with our modified relativistic momentum expression

$$p = -\bar{W}u, \tag{1.33}$$

depending strongly on the scalar vacuum field potential function $\bar{W}: M^4 \to \mathbb{R}$. But if we recall here that our action functional (1.23) was written under the assumption that the particle charge value q is negligible and not exerting the essential influence on the electromagnetic field source, we can make use of the result before obtained in [12, 51, 54], that the vacuum field potential function $\bar{W}: M^4 \to \mathbb{R}$, owing to (1.31)–(1.33), satisfies as $q \to 0$ the dynamical equation

$$d(-\bar{W}u)/dt = -\nabla \bar{W}, \tag{1.34}$$

whose solution will be exactly the expression

$$-\bar{W} = m_0(1-|u|^2)^{-1/2}, \quad m_0 = -\left.\bar{W}\right|_{u=0}. \tag{1.35}$$

Thereby, we have arrived, owing to (1.35) and (1.33), to the almost full coincidence of our result (1.31) for the relativistic Lorentz force with that of (1.6) under the condition $q \to 0$.

The results obtained above and related inferences we will formulate as the following proposition.

Proposition 1.1 *Under the assumption of the negligible influence of a charged point particle q on an external electromagnetic field source a true physically reasonable action functional can be given by expression* (1.22), *being equivalently defined with respect to the rest reference system $\mathcal{K}_r$ in the form* (1.23), (1.24). *The resulting relativistic Lorentz force* (1.31) *coincides almost exactly with that of* (1.6), *obtained from the classical Einstein type action* *functional* (1.1), *but the momentum expression* (1.33) *differs from the classical expression* (1.7), *taking into account the related vacuum field potential interaction energy impact*.

As an important corollary we make the following.

Corollary 1.2 *The Lorentz force expression* (1.31) *should be*, *in due course*, *corrected in the case when the weak charge q influence assumption made above does not hold*.

Remark 1.3 Concerning the infinitesimal relationship (1.25) one can observe that it reflects the Euclidean nature of transformations $\mathbb{R} \ni t \rightleftharpoons \tau \in \mathbb{R}$.

In spite of the results obtained above by means of two different least action principles (1.1) and (1.23), we must claim here that the first one possesses some logical controversies, which may give rise to unpredictable, unexplainable and even nonphysical effects. Amongst these controversies we mention:

i) the definition of Lagrangian function (1.3) as an expression, depending on the external and undefined rest mass parameter with respect to the rest reference system $\mathcal{K}_r$ time $\tau \in \mathbb{R}$, but serving as a variational integrand with respect to the laboratory reference system $\mathcal{K}$ time $t \in \mathbb{R}$;

ii) the least action condition (1.1) is calculated with respect to the fixed boundary conditions at the ends of a time interval $[t_1, t_2] \subset \mathbb{R}$, thereby the resulting dynamics becomes strongly dependent on the chosen laboratory reference system $\mathcal{K}$, which is, following the Feynman arguments [26, 27], physically unreasonable;

iii) the resulting relativistic particle mass and its energy depend only on the particle velocity in the laboratory reference system $\mathcal{K}$, not taking into account the present vacuum field potential energy, exerting not trivial action on the particle motion;

iv) the assumption concerning the negligible influence of a charged point particle on the external electromagnetic field source is also physically inconsistent.

2 The Charged Point Particle Least Action Principle Revisited: The Vacuum Field Theory Approach

2.1 A Free Charged Point Particle in the Vacuum Medium

We start now from the following action functional for a charged point particle q moving with velocity $u := dr/dt \in \mathbb{E}^3$ with respect to a laboratory reference system $\mathcal{K}$:

$$S^{(\tau)} := -\int_{t(\tau_2)}^{t(\tau_1)} \bar{W} dt, \tag{2.1}$$

being defined on the time interval $[t(\tau_1), t(\tau_2)] \subset \mathbb{R}$ by means of a vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$, characterizing the intrinsic properties of the vacuum medium and its interaction with a charged point particle q, jointly with the constraint

$$\langle \dot{\xi}, \dot{\xi} \rangle_{\mathbb{E}^4}^{1/2} = 1, \tag{2.2}$$

where $\xi := (\tau, r) \in \mathbb{E}^4$ is a charged point particle position 4-vector with respect to the proper rest reference system $\mathcal{K}_r$, $\dot{\xi} := d\xi/dt$, $t \in \mathbb{R}$. As the real dynamics of our charged point particle q depends strongly only on the time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ of its own motion subject to the rest reference system $\mathcal{K}_r$, we need to calculate the extremality condition

$$\delta S^{(\tau)} = 0, \qquad \delta r(\tau_1) = 0 = \delta r(\tau_2). \tag{2.3}$$

As action functional (2.1) is equivalent, owing to (2.2) or (1.25), to the following:

$$S^{(\tau)} := -\int_{\tau_2}^{\tau_1} \bar{W}(1 + |\dot{r}|^2)^{1/2} d\tau, \tag{2.4}$$

where, by definition, $\dot{r} := dr/d\tau$, $|\dot{r}|^2 := \langle \dot{r}, \dot{r} \rangle_{\mathbb{E}^3}$, $\tau \in \mathbb{R}$, from (2.4) and (2.3) one easily obtains that

$$p := -\bar{W}\dot{r}(1 + |\dot{r}|^2)^{-1/2}, \qquad dp/d\tau = -\nabla \bar{W}(1 + |\dot{r}|^2)^{1/2}. \tag{2.5}$$

Taking into account once more relationship (1.25) we can rewrite (2.5) equivalently as

$$dp/dt = -\nabla \bar{W}, \quad p := -\bar{W}u. \tag{2.6}$$

If to take into account the dynamic mass definition (1.30), (2.6) turns into the Newton dynamical expression

$$dp/dt = -\nabla \bar{W}, \quad p = mu. \tag{2.7}$$

Having observed now that (2.7) is completely equivalent to (1.34), we obtain right away from (1.35) that the particle mass

$$m = m_0(1 - |u|^2)^{-1/2}, \tag{2.8}$$

where

$$m_0 := -\left. \bar{W} \right|_{u=0} \tag{2.9}$$

is the so-called particle rest mass. Moreover, since the corresponding (2.4) Lagrangian function

$$\mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r}|^2)^{1/2} \tag{2.10}$$

is not degenerate, we can easily construct [1, 2, 12, 22, 32] the related conservative Hamiltonian function

$$\mathcal{H}^{(\tau)} = -(\bar{W}^2 - |p|^2)^{1/2}, \tag{2.11}$$

where $|p|^2 := \langle p, p \rangle_{\mathbb{E}^3}$, satisfying the canonical Hamiltonian equations

$$dr/d\tau = \partial \mathcal{H}^{(\tau)}/\partial p, \qquad dp/d\tau = -\partial \mathcal{H}^{(\tau)}/\partial r \tag{2.12}$$

and conservation conditions

$$d\mathcal{H}^{(\tau)}/dt = 0 = d\mathcal{H}^{(\tau)}/d\tau \tag{2.13}$$

for all $\tau, t \in \mathbb{R}$. Thereby, the quantity

$$\mathcal{E} := (\bar{W}^2 - p^2)^{1/2} \tag{2.14}$$

can be naturally interpreted as the point particle total energy.

It is important to note here that energy expression (2.14) takes into account both kinetic and potential energies, but the particle dynamic mass (2.8) depends only on its velocity, reflecting its free motion in vacuum. Moreover, since the vacuum potential function $\bar{W}: M^4 \to \mathbb{R}$ is not, in general, constant, we claim that the motion of our particle q with respect to the laboratory reference system $\mathcal{K}$ is not, in general, linear and is with not constant velocity,—the situation, which was already discussed before by R. Feynman in [27]. Thus, we obtained the classical relativistic mass dependence on the freely moving particle velocity (2.8), taking into account both the nonconstant vacuum potential function $\bar{W}: M^4 \to \mathbb{R}$ and the particle velocity $u \in \mathbb{E}^3$.

We would also like to mention here that the vacuum potential function $\bar{W}: M^4 \to \mathbb{R}$ itself should be simultaneously found by means of a suitable solution to the Maxwell equation $\partial^2 W/\partial t^2 - \Delta W = \rho$, where $\rho \in \mathbb{R}$ is an ambient charge density and, by definition, $\bar{W}(r(t)) := \lim_{r \to r(t)}. W(r,t)|$, with $r(t) \in \mathbb{E}^3$ being the position of the charged point particle at a time moment $t \in \mathbb{R}$. A more detailed description [51] of the vacuum field potential $W: M^4 \to \mathbb{R}$, characterizing the vacuum medium structure, is given in the Supplement.

We return now to expression (2.1) and rewrite it in the following invariant form

$$S^{(\tau)} = -\int_{s(\tau_1)}^{s(\tau_2)} \bar{W}\langle \dot{\xi}, \dot{\xi}\rangle_{\mathbb{E}^4}^{1/2} ds, \tag{2.15}$$

where, by definition, $s \in \mathbb{R}$ parameterizes the particle world line related with the laboratory reference system $\mathcal{K}$ time parameter $t \in \mathbb{R}$ by means of the Euclidean infinitesimal relationship

$$dt := \langle \dot{\xi}, \dot{\xi}\rangle_{\mathbb{E}^4}^{1/2} ds. \tag{2.16}$$

It is easy to observe that at $s = t \in \mathbb{R}$ functional (2.15) turns into (2.1) and (2.2). The action functional (2.15) is to be supplemented with the boundary conditions

$$\delta\xi(s(\tau_1)) = 0 = \delta\xi(s(\tau_2)), \tag{2.17}$$

which are, obviously, completely equivalent to those of (2.3), since the mapping $\mathbb{R} \ni s \rightleftharpoons t \in \mathbb{R}$, owing to definition (2.16) is one-to-one.

Having calculated the least action condition $\delta S^{(\tau)} = 0$ under constraints (2.17), one easily obtains the same equation (2.6) and relationships (2.8), (2.14) for the particle dynamical mass and its conservative energy, respectively.

2.2 The Charged Point Particle Electrodynamics

We would like to generalize the results obtained above for a free point particle in the vacuum medium for the case of a charged point particle q interacting with external charged point particle q_f, moving with respect to a laboratory reference system $\mathcal{K}$. Within the vacuum field theory approach, devised in [12, 51, 52], it is natural to reduce the formulated problem to that considered above, having introduced the reference system $\mathcal{K}_f$ moving with respect to the reference system $\mathcal{K}$ with the same velocity as that of the external charged point particle q_f. Thus, if the external charged particle q_f, considered with respect to the laboratory reference $\mathcal{K}_f$, will be in rest, the test charged point particle q will be moving with the resulting velocity $u - u_f \in T(\mathbb{E}^3)$, where, by definition, $u := dr/dt$, $u_f := dr_f/dt$, $t \in \mathbb{R}$, are the corresponding velocities of these charged point particles q and q_f with respect to the laboratory reference system $\mathcal{K}$. As a result of these reasonings we can write the following action functional expression

$$S^{(\tau)} = -\int_{s(\tau_1)}^{s(\tau_2)} \bar{W} \langle \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{1/2} ds, \tag{2.18}$$

where, by definition, $\eta_f := (\tau, r - r_f) \in \mathbb{E}^4$ is the charged point particle q position coordinates with respect to the rest reference system $\mathcal{K}_r$ and calculated subject to the introduced laboratory reference system $\mathcal{K}_f$, $s \in \mathbb{R}$ parameterizes the corresponding point particle world line, being infinitesimally related to the time parameter $t \in \mathbb{R}$ as

$$dt := \langle \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{1/2} ds. \tag{2.19}$$

The boundary conditions for functional (2.18) are taken naturally in the form

$$\delta\xi(s(\tau_1)) = 0 = \delta\xi(s(\tau_2)), \tag{2.20}$$

where $\xi = (\tau, r) \in \mathbb{E}^4$. The least action condition $\delta S^{(\tau)} = 0$ jointly with (2.20) gives rise to the following equations:

$$\begin{aligned} P &:= \partial\mathcal{L}^{(\tau)}/\partial\dot{\xi} = -\bar{W}\dot{\eta}_f \langle \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{-1/2}, \\ dP/ds &:= \partial\mathcal{L}^{(\tau)}/\partial\xi = -\nabla_\xi \bar{W} \langle \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{1/2}, \end{aligned} \tag{2.21}$$

where the Lagrangian function equals

$$\mathcal{L}^{(\tau)} := -\bar{W} \langle \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{1/2}. \tag{2.22}$$

Having now defined the charged point particle q momentum $p \in T^*(\mathbb{E}^3)$ as

$$p := -\bar{W}\dot{r} \langle \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{-1/2} = -\bar{W}u \tag{2.23}$$

and the induced external magnetic vector potential $A \in T^*(\mathbb{E}^3)$ as

$$qA := \bar{W}\dot{r}_f \langle \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{-1/2} = \bar{W}u_f, \tag{2.24}$$

we obtain, owing to relationship (2.19), the relativistic Lorentz type force expression

$$dp/dt = qE + qu \times B - q\nabla\langle u, A \rangle_{\mathbb{E}^3}, \tag{2.25}$$

where we denoted, by definition,

$$E := -q^{-1}\nabla\bar{W} - \partial A/\partial t, \qquad B = \nabla \times A, \tag{2.26}$$

being, respectively, the external electric and magnetic fields, acting on the charged point particle q.

The result (2.25) contains the additional Lorentz force component

$$F_c := -q\nabla\langle u, A \rangle_{\mathbb{E}^3}, \tag{2.27}$$

not present in the classical relativistic Lorentz force expressions (1.6) and (1.31), obtained before. Moreover, from (2.23) one obtains that the point particle q momentum

$$p = -\bar{W}u := mu, \tag{2.28}$$

where the particle mass

$$m := -\bar{W} \tag{2.29}$$

does not already coincide with the corresponding classical relativistic relationship of (1.7).

Consider now the least action condition for functional (2.18) at the critical parameter $s = \tau \in \mathbb{R}$:

$$\begin{aligned} \delta S^{(\tau)} &= 0, \qquad \delta r(\tau_1) = 0 = \delta r(\tau_2), \\ S^{(\tau)} &:= -\int_{\tau_1}^{\tau_2} \bar{W}(1 + |\dot{r} - \dot{r}_f|^2_{\mathbb{E}^3})^{1/2} d\tau. \end{aligned} \tag{2.30}$$

The resulting Lagrangian function

$$\mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r} - \dot{r}_f|^2_{\mathbb{E}^3})^{1/2} \tag{2.31}$$

gives rise to the generalized momentum expression

$$P := \partial\mathcal{L}^{(\tau)}/\partial\dot{r} = -\bar{W}(\dot{r} - \dot{r}_f)(1 + |\dot{r} - \dot{r}_f|^2_{\mathbb{E}^3})^{-1/2} := p + qA, \tag{2.32}$$

which makes it possible to construct [1, 2, 22, 32, 53] the corresponding Hamiltonian function as

$$\begin{aligned} \mathcal{H} := \langle P, \dot{r}\rangle_{\mathbb{E}^3} - \mathcal{L}^{(\tau)} &= -(\bar{W}^2 - |p + qA|^2_{\mathbb{E}^3})^{1/2} \\ &\quad - \langle p + qA, qA\rangle_{\mathbb{E}^3}(\bar{W}^2 - |p + qA|^2_{\mathbb{E}^3})^{-1/2}, \end{aligned} \tag{2.33}$$

satisfying the canonical Hamiltonian equations

$$dP/d\tau := \partial\mathcal{H}/\partial r, \qquad dr/d\tau := -\partial\mathcal{H}/\partial r, \tag{2.34}$$

evolving with respect to the proper rest reference system time $\tau \in \mathbb{R}$ parameter. When deriving (2.33) we made use of relationship (2.19) at $s = \tau \in \mathbb{R}$ jointly with definitions (2.23) and (2.24). Since the Hamiltonian function (2.33) is conservative with respect to the evolution parameter $\tau \in \mathbb{R}$, owing to relationship (2.19) at $s = \tau \in \mathbb{R}$ one obtains that

$$d\mathcal{H}/d\tau = 0 = d\mathcal{H}/dt \tag{2.35}$$

for all $t, \tau \in \mathbb{R}$. The obtained results can be formulated as the following proposition.

Proposition 2.1 *The charged point particle electrodynamics, related with the least action principle* (2.18) *and* (2.20), *reduces to the modified Lorentz type force equation* (2.25), *and is equivalent to the canonical Hamilton system* (2.34) *with respect to the proper rest reference system time parameter* $\tau \in \mathbb{R}$. *The corresponding Hamiltonian function* (2.33) *is a conservation law for the Lorentz type dynamics* (2.25), *satisfying the conditions* (2.35) *with respect to both reference systems parameters* $t, \tau \in \mathbb{R}$.

As a corollary, the corresponding energy expression for electrodynamical model (2.25) can be defined as

$$\mathcal{E} := (\bar{W}^2 - |p + qA|^2_{\mathbb{E}^3})^{1/2} + \langle p + qA, qA\rangle_{\mathbb{E}^3}(\bar{W}^2 - |p + qA|^2_{\mathbb{E}^3})^{-1/2}. \tag{2.36}$$

The energy expression (2.36) obtained above is a necessary ingredient for quantizing the relativistic electrodynamics (2.25) of our charged point particle q under the external electromagnetic field influence.

3 A New Hadronic String Model: The Least Action Principle and Relativistic Electrodynamics Analysis Within the Vacuum Field Theory Approach

3.1 A New Hadronic String Model Least Action Formulation

A classical relativistic hadronic string model was first proposed in [5, 30, 46] and deeply studied in [4], making use of the least action principle and related Lagrangian and Hamiltonian formalisms. We will not discuss here this classical string model and will not comment the physical problems accompanying it, especially those related to its diverse quantization versions, but proceed to formulating a new relativistic hadronic string model, constructed by means of the vacuum field theory approach, devised in [12, 51, 52]. The corresponding least action principle is, following [4], formulated as

$$\delta S^{(\tau)} = 0, \qquad S^{(\tau)} := \int_{s(\tau_1)}^{s(\tau_2)} ds \int_{\sigma_1(s)}^{\sigma_2(s)} \bar{W}(x(\xi))(|\dot{\xi}|^2|\xi'|^2 - \langle \dot{\xi}, \xi'\rangle^2_{\mathbb{E}^4})^{1/2} d\sigma \wedge ds, \tag{3.1}$$

where $\bar{W} : M^4 \to \mathbb{R}$ is a vacuum field potential function, characterizing the interaction of the vacuum medium with our string object, the differential 2-form $d\Sigma^{(2)} := (|\dot{\xi}|^2|\xi'|^2 - \langle \dot{\xi}, \xi'\rangle^2_{\mathbb{E}^4})^{1/2} d\sigma \wedge ds = \sqrt[2]{g(\xi)}\, d\sigma \wedge ds$, $g(\xi) := \det(g_{ij}(\xi)\big|_{i,j=\overline{1,2}})$, $|\dot{\xi}|^2 := \langle \dot{\xi}, \dot{\xi}\rangle_{\mathbb{E}^4}$, $|\xi'|^2 := \langle \xi', \xi'\rangle_{\mathbb{E}^4}$, being related with the Euclidean infinitesimal metrics $dz^2 := \langle d\xi, d\xi\rangle_{\mathbb{E}^4} = g_{11}(\xi)d\sigma^2 + g_{12}(\xi)d\sigma ds + g_{21}(\xi)ds d\sigma + g_{22}(\xi)ds^2$ on the string, means [1, 4, 22, 56] the infinitesimal two-dimensional world surface element, parameterized by variables $(\sigma, s) \in \mathbb{E}^2$ and embedded into the 4-dimensional Euclidean space-time with coordinates $\xi := (\tau(\sigma, s), r)) \in \mathbb{E}^4$ subject to the proper rest reference system $\mathcal{K}$, $\dot{\xi} := \partial\xi/\partial s$, $\xi' := \partial\xi/\partial\sigma$ are the corresponding partial derivatives. The related boundary conditions are chosen as

$$\delta\xi(\sigma(s), s) = 0 \tag{3.2}$$

at string parameter $\sigma(s) \in \mathbb{R}$ for all $s \in \mathbb{R}$. The action functional expression is strongly motivated by that constructed for the point particle action functional (2.1):

$$S^{(\tau)} := -\int_{\sigma_1}^{\sigma_2} dl(\sigma) \int_{t(\sigma,\tau_1)}^{t(\sigma,\tau_2)} \bar{W} dt(\tau, \sigma), \tag{3.3}$$

where the laboratory reference time parameter $t(\tau, \sigma) \in \mathbb{R}$ is related to the proper rest string reference system time parameter $\tau \in \mathbb{R}$ by means of the standard Euclidean infinitesimal relationship

$$dt(\tau, \sigma) := (1 + |\dot{r}_\perp|^2(\tau, \sigma))^{1/2} d\tau, \quad |\dot{r}_\perp|^2 := \langle \dot{r}_\perp, \dot{r}_\perp\rangle_{\mathbb{E}^3}, \tag{3.4}$$

with $\sigma \in [\sigma_1, \sigma_2] \subset \mathbb{R}$, being a spatial variable parameterizing the string length measure $dl(\sigma)$ on the real axis $\mathbb{R}$, $\dot{r}_\perp(\tau, \sigma) := \hat{N}\ \dot{r}(\tau, \sigma) \in \mathbb{E}^3$ being the orthogonal to the string velocity component, and

$$\hat{N} := (1 - |r'|^{-2} r' \otimes r'), \quad |r'|^{-2} := \langle r', r' \rangle_{\mathbb{E}^3}^{-1}, \tag{3.5}$$

being the corresponding projector operator in $\mathbb{E}^3$ on the orthogonal to the string direction, expressed for brevity by means of the standard tensor product "$\otimes$" in the Euclidean space $\mathbb{E}^3$. The result of calculation of (3.3) gives rise to the following expression

$$S^{(\tau)} = -\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \bar{W}[(|r'|^2(1 + |\dot{r}|^2) - \langle \dot{r}, r' \rangle_{\mathbb{E}^3}]^{1/2} d\sigma, \tag{3.6}$$

where we made use of the infinitesimal measure representation $dl(\sigma) = \langle r', r' \rangle_{\mathbb{E}^3}^{1/2} d\sigma$, $\sigma \in [\sigma_1, \sigma_2]$. If now to introduce on the string world surface local coordinates $(\sigma, s(\tau, \sigma)) \in \mathbb{E}^2$ and the related Euclidean string position vector $\xi := (\tau, r(\sigma, s)) \in \mathbb{E}^4$, the string action functional reduces equivalently to that of (3.1).

Below we will proceed to Lagrangian and Hamiltonian analyzing the least action conditions for expressions (3.1) and (3.6).

3.2 Lagrangian and Hamiltonian Analysis

First we will obtain the corresponding to (3.1) Euler equations with respect to the special [4, 22] internal conformal variables $(\sigma, s) \in \mathbb{E}^2$ on the world string surface, with respect to which the metrics on it becomes equal to $dz^2 = |\xi'|^2 d\sigma^2 + |\dot{\xi}|^2 ds^2$, where $\langle \xi', \dot{\xi} \rangle_{\mathbb{E}^4} = 0 = |\xi'|^2 - |\dot{\xi}|^2$, and the corresponding infinitesimal world surface measure $d\Sigma^{(2)}$ becomes $d\Sigma^{(2)} = |\xi'||\dot{\xi}| d\sigma \wedge ds$. As a result of simple calculations one finds the linear second order partial differential equation

$$\partial(\bar{W}\dot{\xi})/\partial s + \partial(\bar{W}\xi')/\partial\sigma = |\xi'||\dot{\xi}|\partial\bar{W}/\partial\sigma \tag{3.7}$$

under the suitably chosen boundary conditions

$$\xi' - \dot{\xi}\dot{\sigma} = 0 \tag{3.8}$$

for all $s \in \mathbb{R}$. It is interesting to mention that (3.7) is of elliptic type, contrary to the case considered before in [4]. This is, evidently, owing to the fact that the resulting metrics on the string world surface is Euclidean, as we took into account that the real string motion is, in reality, realized with respect to its proper rest reference system $\mathcal{K}_r$, being not dependent on the string motion observation data, measured with respect to any external laboratory reference system $\mathcal{K}$.

The differential equation (3.7) strongly depends on the vacuum field potential function $\bar{W}: M^4 \to \mathbb{R}$, which, as a function of the Minkovski 4-vector variable $x := (t(\sigma, s), r) \in M^4$ of the laboratory reference system $\mathcal{K}$, should be expressed by means of the infinitesimal relationship (3.4) in the following form:

$$dt = \langle \hat{N}\partial\xi/\partial\tau, \hat{N}\partial\xi/\partial\tau \rangle^{1/2} \left(\frac{\partial\tau}{\partial s} ds + \frac{\partial\tau}{\partial\sigma} d\sigma \right), \tag{3.9}$$

defined on the string world surface. The function $\bar{W}: M^4 \to \mathbb{R}$ itself should be simultaneously found by means of a suitable solution to the Maxwell equation $\partial^2 W/\partial t^2 - \Delta W = \rho$, where $\rho \in \mathbb{R}$ is an ambient charge density and, by definition, $\bar{W}(r(t)) := \lim_{r \to r(t)}. W(r,t)|$, with $r(t) \in \mathbb{E}^3$ being the position of the string element with the proper rest reference coordinates $(\sigma, \tau) \in \mathbb{E}^2$ at a time moment $t = t(\sigma, \tau) \in \mathbb{R}$.

We proceed now to constructing the dynamical Euler equations for our string model, making use of action functional (3.6). It is easy to calculate that the generalized momentum

$$p := \partial \mathcal{L}^{(\tau)}/\partial \dot{r} = \frac{-\bar{W}(|r'|^2\dot{r} - r'\langle r', \dot{r}\rangle_{\mathbb{E}^3})}{[|r'|^2(|\dot{r}|^2+1) - \langle r', \dot{r}\rangle_{\mathbb{E}^3}^2]^{1/2}} = \frac{-\bar{W}(|r'|^2\hat{N}\dot{r})}{[|r'|^2(|\dot{r}|^2+1) - \langle r', \dot{r}\rangle_{\mathbb{E}^3}^2]^{1/2}} \tag{3.10}$$

satisfies the dynamical equation

$$dp/d\tau := \delta \mathcal{L}^{(\tau)}/\delta r = -[|r'|^2(|\dot{r}|^2+1) - \langle r', \dot{r}\rangle_{\mathbb{E}^3}^2]^{1/2}\nabla \bar{W} - \frac{\partial}{\partial \sigma}\left\{\frac{\bar{W}(1+|\dot{r}|^2\hat{T})r'}{[|r'|^2 + |\dot{r}|^2\langle r', \hat{T}r'\rangle]^{1/2}}\right\}, \tag{3.11}$$

where we denoted by

$$\mathcal{L}^{(\tau)} := -\bar{W}[(|r'|^2(1+|\dot{r}|^2) - \langle \dot{r}, r'\rangle_{\mathbb{E}^3}^2]^{1/2} = -\bar{W}[|r'|^2 + |\dot{r}|^2\langle r', \hat{T}r'\rangle]^{1/2} \tag{3.12}$$

the corresponding Lagrangian function and by

$$\hat{T} := 1 - |\dot{r}|^{-2}\dot{r} \otimes \dot{r}, \quad |\dot{r}|^{-2} := \langle \dot{r}, \dot{r}\rangle_{\mathbb{E}^3}^{-2}, \tag{3.13}$$

the related dynamic projector operator in $\mathbb{E}^3$. The Lagrangian function is degenerate [4, 22], satisfying the obvious identity

$$\langle p, r'\rangle_{\mathbb{E}^3} = 0 \tag{3.14}$$

for all $\tau \in \mathbb{R}$. Concerning the Hamiltonian formulation of the dynamics (3.11) we construct the corresponding Hamiltonian functional as

$$\begin{aligned} \mathcal{H} &:= \int_{\sigma_1}^{\sigma_2} (\langle p, \dot{r}\rangle_{\mathbb{E}^3} - \mathcal{L}^{(\tau)})d\sigma \\ &= \int_{\sigma_1}^{\sigma_2} \bar{W}r'^{,2}[(|r'|^2(1+|\dot{r}|^2) - \langle \dot{r}, r'\rangle_{\mathbb{E}^3}^2]^{1/2}d\sigma \\ &= \int_{\sigma_1}^{\sigma_2} [\bar{W}^2|r'|^2 - p^2]^{1/2}d\sigma, \end{aligned} \tag{3.15}$$

satisfying the canonical equations

$$dr/d\tau := \delta\mathcal{H}/\delta p, \qquad dp/d\tau := -\delta\mathcal{H}/\delta r, \tag{3.16}$$

where

$$d\mathcal{H}/d\tau = 0, \tag{3.17}$$

holding only with respect to the proper rest reference system $\mathcal{K}_r$ time parameter $\tau \in \mathbb{R}$. Now making use of identity (3.14) the Hamiltonian functional (3.15) can be equivalently represented in the symbolic form as

$$\mathcal{H} = \int_{\sigma_1}^{\sigma_2} |\bar{W} r' \pm ip|_{\mathbb{E}^3} d\sigma, \tag{3.18}$$

where $i := \sqrt{-1}$. Moreover, concerning the result obtained above we need to mention here that one can not construct the suitable Hamiltonian function expression and relationship of type (3.17) with respect to the laboratory reference system $\mathcal{K}$, since expression (3.18) is not defined on the whole for a separate laboratory time parameter $t \in \mathbb{R}$ locally dependent both on the spatial parameter $\sigma \in \mathbb{R}$ and the proper rest reference system time parameter $\tau \in \mathbb{R}$.

Thereby, one can formulate the following proposition.

Proposition 3.1 *The hadronic string model* (3.1) *allows*, *on the related world surface*, *the conformal local coordinates*, *with respect to which the resulting dynamics is described by means of the linear second order elliptic equation* (3.7). *Subject to the proper rest reference system Euclidean coordinates the corresponding dynamics is equivalent to the canonical Hamiltonian equations* (3.16) *with Hamiltonian functional* (3.15).

We proceed now to construct the action functional expression for a charged string under an external magnetic field, generated by a point velocity charged particle q_f, moving with some velocity $u_f := dr_f/dt \in \mathbb{E}^3$ subject to a laboratory reference system $\mathcal{K}$. To solve this problem we make use of the trick of Sect. 2 above, passing to the proper rest reference system $\mathcal{K}_r$ with respect to the relative reference system $\mathcal{K}_f$, moving with velocity $u_f \in \mathbb{E}^3$. As a result of this reasoning we can write down the action functional:

$$S^{(\tau)} = -\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \bar{W}[|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r'\rangle_{\mathbb{E}^3}^2]^{1/2} d\sigma, \tag{3.19}$$

giving rise to the following dynamical equation

$$\begin{aligned} dP/d\tau := \delta\mathcal{L}^{(\tau)}/\delta r = &-[|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r'\rangle_{\mathbb{E}^3}^2]^{1/2} \nabla \bar{W} \\ &+ \frac{\partial}{\partial\sigma}\left\{\frac{\bar{W}(1 + |\dot{r} - \dot{r}_f|^2 \hat{T}_f) r'}{[|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r'\rangle_{\mathbb{E}^3}^2]^{1/2}}\right\}, \end{aligned} \tag{3.20}$$

where the generalized momentum

$$P := \frac{-\bar{W}[|r'|^2 \hat{N}(\dot{r} - \dot{r}_f)]}{[|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r'\rangle_{\mathbb{E}^3}^2]^{1/2}} \tag{3.21}$$

and the projection operator in $\mathbb{E}^3$

$$\hat{T}_f := 1 - |\dot{r} - \dot{r}_f)|^{-2} \qquad (\dot{r} - \dot{r}_f) \otimes (\dot{r} - \dot{r}_f). \tag{3.22}$$

Having defined by

$$p := \frac{-\bar{W}(|r'|^2 \hat{N} \dot{r})}{[|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r'\rangle_{\mathbb{E}^3}^2]^{1/2}} \tag{3.23}$$

the local string momentum and by

$$qA := \frac{\bar{W}(|r'|^2 \hat{N} \dot{r}_f)}{[|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f, r'\rangle^2_{\mathbb{E}^3}]^{1/2}} \tag{3.24}$$

the external vector magnetic potential, (3.20) reduces to

$$\begin{aligned} dp/d\tau &= q\dot{r} \times B - q\nabla\langle A, \dot{r}\rangle_{\mathbb{E}^3} - q\frac{\partial A}{\partial \tau} \\ &\quad - [|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f, r'\rangle^2_{\mathbb{E}^3}]^{1/2}\nabla\bar{W} \\ &\quad + \frac{\partial}{\partial\sigma}\left\{\frac{\bar{W}(1+|\dot{r}-\dot{r}_f|^2\hat{T}_f)r'}{[|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f, r'\rangle^2_{\mathbb{E}^3}]^{1/2}}\right\}, \end{aligned} \tag{3.25}$$

where $q \in \mathbb{R}$ is a charge density, distributed along the string length, $B := \nabla \times A$ means the external magnetic field, acting on the string. The expression, defined as

$$\begin{aligned} E &:= -q\frac{\partial A}{\partial \tau} - [|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f, r'\rangle^2_{\mathbb{E}^3}]^{1/2}\nabla\bar{W} \\ &\quad + \frac{\partial}{\partial\sigma}\left\{\frac{\bar{W}(1+|\dot{r}-\dot{r}_f|^2\hat{T}_f)r'}{[|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f, r'\rangle^2_{\mathbb{E}^3}]^{1/2}}\right\}, \end{aligned} \tag{3.26}$$

similar to the charged point particle case, models a related electric field, exerted on the string by the external electric charge q_f. Making use of the standard scheme, one can derive, as above, the Hamiltonian interpretation of dynamical equations (3.20), but which will not be here discussed.

4 Conclusion

Based on the vacuum field theory approach, devised recently in [12, 51, 52], we revisited the alternative charged point particle and hadronic string electrodynamics models, having succeeded in treating their Lagrangian and Hamiltonian properties. The obtained results were compared with classical ones, owing to which a physically motivated choice of a true model was argued. Another important aspect of the developed vacuum field theory approach consists in singling out the decisive role of the related rest reference system $\mathcal{K}_r$, with respect to which the relativistic object motion, in reality, realizes. Namely, with respect to the proper rest reference system evolution parameter $\tau \in \mathbb{R}$ all of our electrodynamics models allow both the Lagrangian and Hamiltonian physically reasonable formulations, suitable for the canonical procedure. The deeper physical nature of this fact remains, up today, as we assume, not enough understood. We would like to recall here only very interesting reasonings by R. Feynman who argued in [26, 27] that the relativistic expressions have physical sense only with respect to the proper rest reference systems. In a sequel of our work we plan to analyze our relativistic electrodynamic models subject to their quantization and make a step toward the related vacuum quantum field theory of infinite many particle systems.

5 Supplement: The Maxwell Electromagnetism Theory

5.1 The Vacuum Field Theory Look and Interpretation

We start from the following field theoretical model [12] of the microscopic vacuum medium structure, considered as some physical reality imbedded into the standard three-dimensional Euclidean space reference system marked by three spatial coordinates $r \in \mathbb{E}^3$, endowed, as before, with the standard scalar product $\langle \cdot, \cdot \rangle_{\mathbb{E}^3}$, and parameterized by means of the scalar temporal parameter $t \in \mathbb{R}$. First we will describe the physical vacuum medium endowing it with an everywhere smooth enough four-vector potential function $(W, A) : M^4 \to \mathbb{R} \times \mathbb{E}^3$, defined in the Minkovski space M^4 and naturally related to light propagation properties. The material objects, imbedded into the vacuum medium, we will model (classically here) by means of the scalar charge density function $\rho : M^4 \to \mathbb{R}$ and the vector current density $J : M^4 \to \mathbb{E}^3$, being also everywhere smooth enough functions.

(i) The *first* field theory principle regarding the vacuum we accept is formulated as follows: the four-vector function $(W, A) : M^4 \to \mathbb{R} \times \mathbb{E}^3$ satisfies the standard Lorentz type continuity relationship

$$\frac{1}{c}\frac{\partial W}{\partial t} + \langle \nabla, A \rangle_{\mathbb{E}^3} = 0, \tag{5.1}$$

where, by definition, $\nabla := \partial/\partial r$ is the usual gradient operator.

(ii) The *second* field theory principle we accept is a dynamical relationship on the scalar potential component $W : M^4 \to \mathbb{R}$:

$$\frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} - \nabla^2 W = \rho, \tag{5.2}$$

assuming the linear law of the small vacuum uniform and isotropic perturbation propagations in the space-time, understood here, evidently, as a first (linear) approximation in the case of weak enough fields.

(ii) The *third* principle is similar to the first one and means simply the continuity condition for the density and current density functions:

$$\partial\rho/\partial t + \langle \nabla, J \rangle_{\mathbb{E}^3} = 0. \tag{5.3}$$

We need to note here that the vacuum field perturbations velocity parameter $c > 0$, used above, coincides with the vacuum light velocity, as we are trying to derive successfully from these first principles the well-known Maxwell electromagnetism field equations, to analyze the related Lorentz forces and special relativity relationships. To do this, we first combine (5.1) and (5.2):

$$\frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = -\left\langle \nabla, \frac{1}{c}\frac{\partial A}{\partial t} \right\rangle_{\mathbb{E}^3} = \left\langle \nabla, \nabla W \right\rangle_{\mathbb{E}^3} + \rho,$$

whence

$$\left\langle \nabla, -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla W \right\rangle_{\mathbb{E}^3} = \rho. \tag{5.4}$$

Having put, by definition,

$$E := -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla W, \tag{5.5}$$

we obtain the first material Maxwell equation

$$\langle \nabla, E\rangle_{\mathbb{E}^3} = \rho \tag{5.6}$$

for the electric field $E : M^4 \to \mathbb{E}^3$. Having now applied the rotor-operation $\nabla\times$ to expression (5.5) we obtain the first Maxwell field equation

$$\frac{1}{c}\frac{\partial B}{\partial t} - \nabla \times E = 0 \tag{5.7}$$

on the magnetic field vector function $B : M^4 \to \mathbb{E}^3$, defined as

$$B := \nabla \times A. \tag{5.8}$$

To derive the second Maxwell field equation we will make use of (5.8), (5.1) and (5.5):

$$\begin{aligned}\nabla \times B &= \nabla \times (\nabla \times A) = \nabla\langle \nabla, A\rangle_{\mathbb{E}^3} - \nabla^2 A \\ &= \nabla\left(-\frac{1}{c}\frac{\partial W}{\partial t}\right) - \nabla^2 A = \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla W - \frac{1}{c}\frac{\partial A}{\partial t} + \frac{1}{c}\frac{\partial A}{\partial t}\right) - \nabla^2 A \\ &= \frac{1}{c}\frac{\partial E}{\partial t} + \left(\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A\right).\end{aligned} \tag{5.9}$$

We have from (5.5), (5.6) and (5.3) that

$$\left\langle \nabla, \frac{1}{c}\frac{\partial E}{\partial t}\right\rangle_{\mathbb{E}^3} = \frac{1}{c}\frac{\partial \rho}{\partial t} = -\frac{1}{c}\langle \nabla, J\rangle_{\mathbb{E}^3},$$

or

$$\left\langle \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla\left(\frac{1}{c}\frac{\partial W}{\partial t}\right) + \frac{1}{c}J\right\rangle_{\mathbb{E}^3} = 0. \tag{5.10}$$

Now making use of (5.1), from (5.10) we obtain that

$$\begin{aligned}&\left\langle \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla\left(\frac{1}{c}\frac{\partial W}{\partial t}\right) + \frac{1}{c}J\right\rangle_{\mathbb{E}^3} = \left\langle \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla\langle \nabla, A\rangle_{\mathbb{E}^3} + \frac{1}{c}J\right\rangle_{\mathbb{E}^3} \\ &= \left\langle \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla^2 A + \nabla \times (\nabla \times A) + \frac{1}{c}J\right\rangle_{\mathbb{E}^3} \\ &= \left\langle \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla^2 A + \frac{1}{c}J\right\rangle_{\mathbb{E}^3} = 0.\end{aligned} \tag{5.11}$$

Thereby, (5.11) yields

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{1}{c}(J + \nabla \times S) \tag{5.12}$$

for some smooth vector function $S : M^4 \to \mathbb{E}^3$. Here we need to note that continuity equation (5.3) is defined, concerning the current density vector $J : M^4 \to \mathbb{R}^3$, up to a vorticity expression, that is $J \simeq J + \nabla \times S$ and (5.12) can finally be rewritten as

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{1}{c}J. \tag{5.13}$$

Having substituted (5.13) into (5.9) we obtain the second Maxwell field equation

$$\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{1}{c} J. \tag{5.14}$$

In addition, from (5.8) one also finds the magnetic no-charge relationship

$$\langle \nabla, B \rangle_{\mathbb{E}^3} = 0. \tag{5.15}$$

Thus, we have derived all the Maxwell electromagnetic field equations from our three main principles (5.1), (5.2) and (5.3). The success of our undertaking will be more impressive if we adapt our results to those following from the well-known relativity theory in the case of point charges or masses. Below we will try to demonstrate the corresponding derivations based on some completely new physical conceptions of the vacuum medium first discussed in [12, 54].

Corollary 5.1 *It is interesting to analyze a partial case of the first field theory vacuum principle* (5.1) *when the following local conservation law for the scalar potential field function* $W : M^4 \to \mathbb{R}$ *holds*:

$$\frac{d}{dt}\int_{\Omega_t} W d^3 r = 0, \tag{5.16}$$

where $\Omega_t \subset \mathbb{E}^3$ *is any open domain in space* $\mathbb{E}^3$ *with the smooth boundary* $\partial\Omega_t$ *for all* $t \in \mathbb{R}$ *and* $d^3 r$ *is the standard volume measure in* $\mathbb{E}^3$ *in a vicinity of the point* $r \in \Omega_t$. *Having calculated expression* (5.16) *we obtain the following equivalent continuity equation*

$$\frac{1}{c}\frac{\partial W}{\partial t} + \left\langle \nabla, \frac{v}{c} W \right\rangle_{\mathbb{E}^3} = 0, \tag{5.17}$$

where $\nabla := \nabla_r$ *is*, *as above*, *the gradient operator and* $v := dr/dt$ *is the velocity vector of a vacuum medium perturbation at point* $r \in \mathbb{E}^3$ *carrying the field potential quantity* W. *Comparing now equations* (5.1), (5.17) *and using* (5.3) *we can make the suitable very important identifications*:

$$A = \frac{v}{c} W, \qquad J = \rho v, \tag{5.18}$$

well known from the classical electrodynamics [38] *and superconductivity theory* [26, 36]. *Thus*, *we are faced with a new physical interpretation of the conservative electromagnetic field theory when the vector potential* $A : M^4 \to \mathbb{E}^3$ *is completely determined via expression* (5.18) *by the scalar field potential function* $W : M^4 \to \mathbb{R}$. *It is also evident that all the Maxwell electromagnetism filed equations derived above hold as well in the case* (5.18), *as it was first demonstrated in* [12].

Consider now the conservation equation (5.16) jointly with the related integral "vacuum momentum" conservation condition

$$\frac{d}{dt}\int_{\Omega_t} \left(\frac{W v}{c^2}\right) d^3 r = 0, \qquad \Omega_t|_{t=0} = \Omega_0, \tag{5.19}$$

where, as above, $\Omega_t \subset \mathbb{E}^3$ is for any time $t \in \mathbb{R}$ an open domain with the smooth boundary $\partial\Omega_t$, whose evolution is governed by the equation

$$dr/dt = v(r, t) \tag{5.20}$$

for all $r \in \Omega_t$ and $t \in \mathbb{R}$, as well as by the initial state of the boundary $\partial\Omega_0$. As a result of relation (5.19) one obtains the new continuity equation

$$\frac{d(vW)}{dt} + vW\langle\nabla, v\rangle_{\mathbb{E}^3} = 0. \tag{5.21}$$

Now making use of (5.17) in the equivalent form

$$\frac{dW}{dt} + W\langle\nabla, v\rangle_{\mathbb{E}^3} = 0,$$

we finally obtain a very interesting local conservation relationship

$$dv/dt = 0 \tag{5.22}$$

on the vacuum matter perturbations velocity $v = dr/dt$, which holds for all values of the time parameter $t \in \mathbb{R}$. As it is easy to observe, the obtained relationship completely coincides with the well-known hydrodynamic equation [43] of ideal compressible liquid without any external exertion, that is, any external forces and field "pressure" are equally identical to zero. We received a natural enough result where the propagation velocity of the vacuum field matter is constant and equals exactly $v = c$, that is the light velocity in the vacuum, if to take into account the starting wave equation (5.2) owing to which the small vacuum field matter perturbations propagate in the space with the light velocity.

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